

New features of some proton-neutron collective states

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Abstract

Using a schematic solvable many-body Hamiltonian, one studies a new type of proton-neutron excitations within a time dependent variational approach. Classical equations of motion are linearized and subsequently solved analytically. The harmonic state energy is compared with the energy of the first excited state provided by diagonalization as well as with the energies obtained by a renormalized RPA and a boson expansion procedure. The new collective mode describes a wobbling motion, in the space of isospin, and collapses for a particle-particle interaction strength which is much larger than the physical value. A suggestion for the description of the system in the second nuclear phase is made. We identified the transition operators which might excite the new mode from the ground state.

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I. INTRODUCTION

One of the most exciting subject in theoretical nuclear physics is the double beta decay, especially due to the neutrino-less ($0\nu\beta\beta$) process [1–4]. Indeed, its discovery would answer a fundamental question whether neutrino is a Majorana or a Dirac particle. The theories devoted to the description of this process suffer of the lack of reliable tests for the nuclear matrix elements. One possibility to overcome such difficulties would be to use the matrix elements which describe realistically the rate of $2\nu\beta\beta$ decay. In this context many theoretical work have been focussed on $2\nu\beta\beta$ process. Most formalisms are based on the proton-neutron quasiparticle random phase approximation (pnQRPA) which includes the particle-particle (pp) channel in the two body interaction. Since such an interaction is not considered in the mean field equations the approach fails at a critical value of the interaction strength, g_{pp} . Before this value is reached, the Gamow-Teller transition amplitude (M_{GT}) is decreasing rapidly and after a short interval is becoming equal to zero. The experimental data for this amplitude is reached for a value of g_{pp} close to that one which vanishes M_{GT} and also close to the critical value. Along the time, the instability of the pnQRPA ground state was considered in different approaches. The first formalism devoted to this feature includes anharmonicities through the boson expansion technique [5–8]. Another method is the renormalized pnQRPA procedure (pnQRRPA) [9] which keeps the harmonic picture but the actual boson is renormalized by effects coming from the terms of the commutators algebra, which are not taken into account in the standard pnQRPA approach.

In a previous paper [10] we have proved that the pnQRRPA procedure does not include the additional effects in an consistent way. Indeed, if the commutators of two quasiparticle operators involves the average of monopole terms then these terms should be considered also in the commutators of the scattering terms. If one does so, new degrees of freedom are switched on and a new pnQRPA boson can be defined. This contains, besides the standard two quasiparticle operators, the proton-neutron quasiparticles scattering terms. If the amplitude of the scattering term is dominant comparing it to the other amplitudes, the

pnQRRPA phonon describes a new nuclear state.

The aim of this paper is to show that such a mode appears in a natural way within a time dependent treatment. The present approach points out new properties of the new proton-neutron collective mode. We use a schematic many-body Hamiltonian which for a single j -shell is exactly solvable. In this way the approximations might be judged by comparing the predictions of the actual model with the corresponding exact results. Since the semi-classical treatment is the proper way to determine the mean field, one expects that the present approach is suitable to account for ground state correlations in a consistent way and therefore some of the drawbacks mentioned in a previous publication [10], like the breaking down of the fully renormalized RPA before the standard RPA breaks down, are removed. To understand better the virtues of the present model we compare its predictions with the results obtained in a renormalized RPA approach and a boson expansion formalism. Since the semi-classical methods have, sometimes, intuitive grounds we aim at obtaining a clear interpretation for the new proton-neutron mode. It is well known that the breaking down of the RPA approach is associated to a phase transition. In this respect the semi-classical formalism is a suitable framework to define the nuclear phases which are bridged by the Goldstone mode. Above arguments justify our option for a semi-classical treatment and also sketch a set of expectations.

This project is achieved according to the following plan: In Section II, we describe the model Hamiltonian. The main features of the fully renormalized RPA approach, presented in a previous paper, are briefly reviewed. A time dependent variational principle is formulated in connection with a truncated quasiparticle Hamiltonian, in Section III. This Hamiltonian is the term of the model Hamiltonian which determines the equations of motion for the quasiparticle proton-neutron scattering terms, in the de-coupling regime. The classical equations of motion and their solutions are presented in Section IV. The new pn collective mode is alternatively described through the renormalized RPA approach and boson expansion formalism in Section V. Numerical results are analyzed in Section VI while the final conclusions are given in Section VII.

II. THE MODEL HAMILTONIAN. BRIEF REVIEW OF FRN-RPA

Since we are not going to describe realistically some experimental data but to stress on some specific features of a heterogeneous many nucleon system with proton-neutron interaction, we consider a schematic Hamiltonian which is very often [11,12] used to study the single and double beta Fermi transitions:

$$\begin{aligned}
H = & \sum_{jm} (\varepsilon_{pj} - \lambda_p) c_{pj}^\dagger c_{pj} + \sum_{jm} (\varepsilon_{nj} - \lambda_n) c_{nj}^\dagger c_{nj} \\
& - \frac{G_p}{4} \sum_{jm, j'm'} c_{pj}^\dagger c_{pj}^\dagger \widetilde{c_{pj}} c_{pj'm'}^\dagger c_{pj'm'} - \frac{G_n}{4} \sum_{jm, j'm'} c_{nj}^\dagger c_{nj}^\dagger \widetilde{c_{nj}} c_{nj'm'}^\dagger c_{nj'm'} \\
& + \chi \sum_{jm, j'm'} c_{pj}^\dagger c_{nj}^\dagger \widetilde{c_{nj}} c_{nj'm'}^\dagger c_{pj'm'} - \chi_1 \sum_{jm, j'm'} c_{pj}^\dagger c_{nj}^\dagger \widetilde{c_{nj}} c_{nj'm'}^\dagger c_{pj'm'}.
\end{aligned} \tag{2.1}$$

$c_{\tau jm}^\dagger (c_{\tau jm})$ denotes the creation (annihilation) of a $\tau (= p, n)$ nucleon in a spherical shell model state $|\tau; nljm\rangle = |\tau jm\rangle$ with τ taking the values p for protons and n for neutrons, respectively. The time reversed state corresponding to $|\tau jm\rangle$ is $|\widetilde{\tau jm}\rangle = (-)^{j-m} |\tau j - m\rangle$

For what follows it is useful to introduce the quasiparticle (qp) representation, defined by the Bogoliubov-Valatin (BV) transformation:

$$\begin{aligned}
a_{pj}^\dagger &= U_{pj} c_{pj}^\dagger - V_{pj} \widetilde{c_{pj}}, \quad a_{pj} = U_{pj} c_{pj} - V_{pj}^* \widetilde{c_{pj}}^\dagger, \\
a_{nj}^\dagger &= U_{nj} c_{nj}^\dagger - V_{nj} \widetilde{c_{nj}}, \quad a_{nj} = U_{nj} c_{nj} - V_{nj}^* \widetilde{c_{nj}}^\dagger.
\end{aligned} \tag{2.2}$$

which quasi-diagonalizes the first four terms, i.e in the new representation they are replaced by a set of independent quasiparticles of energies:

$$E_\tau = \sqrt{(\epsilon_\tau - \lambda_\tau)^2 + \Delta_\tau^2}. \tag{2.3}$$

In the new qp representation, the model Hamiltonian, denoted by H_q , describes a set of independent quasiparticles, interacting among themselves through a two body interaction determined by the images of the χ and χ_1 terms through the BV transformation.

Various many-body approaches have been tested by using not the qp image of H but another Hamiltonian derived from H by ignoring the scattering qp terms:

$$\begin{aligned} B^\dagger(jpn) &= \sum_m a_{pj}^\dagger a_{njm}, \\ B(jpn) &= \sum_m a_{njm}^\dagger a_{pj}. \end{aligned} \quad (2.4)$$

and restricting the space of single particle states to a single j -state. Thus, the model Hamiltonian contains, besides the terms for the qp independent motion, a two body term which is quadratic in the two quasiparticle operators A^\dagger, A :

$$A^\dagger(jpn) = \sum_m a_{pj}^\dagger a_{\widetilde{njm}}, \quad A(jpn) = (A^\dagger(jpn))^\dagger. \quad (2.5)$$

In a previous publication [10], we showed that going beyond the quasiparticle random phase approximation (pnQRPA) through a renormalization procedure, a new degree of freedom is switched on, which results in having a renormalized pnQRPA boson operator as a superposition of the operators $A^\dagger(jpn), A(jpn)$ and scattering terms $B^\dagger(jpn), B(jpn)$. This picture differs from the standard $pnQRPA$ approach, where the boson operators involve only the operators A^\dagger and A , and is conventionally called as fully renormalized RPA ($frn - RPA$). Obviously, when the amplitudes of scattering terms are dominant, one deals with a new kind of collective pn excitation.

In order to define clearly the distinct features of the new proton-neutron (pn) mode revealed in the present paper a brief description of the results obtained in a previous publication [10] is necessary. The equations of motion associated to the many-body Hamiltonian, written in terms of quasiparticle operators, are determined by the commutators algebra of the two quasiparticle (A^\dagger, A) and scattering (B^\dagger, B) operators defined by eqs. (2.5) and (2.4) respectively. Within the $frn - RPA$, the exact commutators are approximated as follows:

$$\begin{aligned} [A(jpn), A^\dagger(jpn)] &= C_{jpn}^{(1)}, \\ [B(jpn), B^\dagger(jpn)] &= C_{jpn}^{(2)}, \\ [A(jpn), B^\dagger(jpn)] &= [A(jpn), B(jpn)] = 0. \end{aligned} \quad (2.6)$$

The terms $C_{jpn}^{(1)}, C_{jpn}^{(2)}$ appearing in the r.h.s. of the above equations are the averages of the corresponding exact commutators, on the correlated ground state $|0\rangle$:

$$C_{jpn}^{(1)} = \langle 0 | 1 - \hat{N}_{jn} - \hat{N}_{jp} | 0 \rangle, \quad C_{jpn}^{(2)} = \langle 0 | \hat{N}_{jn} - \hat{N}_{jp} | 0 \rangle. \quad (2.7)$$

with $\hat{N}_{j\tau}$ standing for the τ (=p,n) quasiparticle number operator in the shell j . The normalized operators

$$\begin{aligned} \bar{A}^\dagger(jpn) &= \frac{1}{\sqrt{C_{jpn}^{(1)}}} A^\dagger(jpn), \quad \bar{A}(jpn) = \left(\bar{A}^\dagger(jpn) \right)^\dagger, \\ \bar{B}^\dagger(jpn) &= \frac{1}{\sqrt{|C_{jpn}^{(2)}|}} B^\dagger(jpn), \quad \bar{B}(jpn) = \left(\bar{B}^\dagger(jpn) \right)^\dagger, \end{aligned} \quad (2.8)$$

satisfy bosonic commutation relations and thereby their equations of motion are linear:

$$\left[H_q, \begin{pmatrix} \bar{A}^\dagger(jpn) \\ \bar{A}(jpn) \\ \bar{B}^\dagger(jpn) \\ \bar{B}(jpn) \end{pmatrix} \right] = \sum_{j,j'} T^{j,j'} \begin{pmatrix} \bar{A}^\dagger(jpn) \\ \bar{A}(jpn) \\ \bar{B}^\dagger(jpn) \\ \bar{B}(jpn) \end{pmatrix}. \quad (2.9)$$

The matrix $T^{j,j'}$ depends on the U and V coefficients as well as on the strengths χ, χ_1 of the two body interactions. The $frn - RPA$ approach defines a linear combination of the basic operators $\bar{A}^\dagger(jpn), \bar{A}(jpn), \bar{B}^\dagger(jpn), \bar{B}(jpn)$,

$$\Gamma^\dagger = \sum_j \left[X(j) \bar{A}^\dagger(jpn) + Z(j) D^\dagger(jpn) - Y(j) \bar{A}(jpn) - W(j) D(jpn) \right], \quad (2.10)$$

so that the following commutation relations with its hermitian conjugate operator and the model Hamiltonian hold:

$$[\Gamma, \Gamma^\dagger] = 1, \quad (2.11)$$

$$[H_q, \Gamma^\dagger] = \omega \Gamma^\dagger. \quad (2.12)$$

The operators $D^\dagger(jpn)$ are identical with $\bar{B}^\dagger(jpn)$ or $\bar{B}(jpn)$ depending on whether the sign of $C_{jpn}^{(2)}$ is plus or minus. The equation (2.12) provides a set of homogeneous equations- called the $frn - RPA$ equations- for the amplitudes X, Y, Z, W :

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ -\mathcal{B} & -\mathcal{A} \end{pmatrix} \begin{pmatrix} X \\ Z \\ Y \\ W \end{pmatrix} = \omega \begin{pmatrix} X \\ Z \\ Y \\ W \end{pmatrix}, \quad (2.13)$$

while the equation (2.11) yields the normalization equation

$$\sum_j (X^2(j) + Z^2(j) - Y^2(j) - W^2(j)) = 1. \quad (2.14)$$

The $frn - RPA$ matrices depend on the renormalization constants $C^{(1)}, C^{(2)}$ which, at their turn, depend on the phonon amplitudes. Therefore, the equations (2.12) and (2.7) should be self-consistently solved.

In ref. [10] the $frn - RPA$ equations have been solved both for a proton-neutron dipole-dipole interaction, needed for the description of the double beta Gamow-Teller decay and for a proton-neutron monopole-monopole interaction used in the calculation of the rates of the double beta Fermi decay. Equations obtained in the two cases have some common features which, for what follows, are worth being enumerated.

- 1) The dimension of the $frn - RPA$ matrix is twice as large as that of the standard RPA and consequently new solutions show up.
- 2) The solutions characterized by that the largest phonon amplitude is of type Z define a new class of proton-neutron excitations.
- 3) Due to the attractive character of the two body interaction in the particle-particle (pp) channel, the lowest new state has an energy which is smaller than the minimal absolute value of the relative energy of the proton and neutron quasiparticle partner states, related by the operators $B^\dagger(jpn), B(jpn)$.
- 4) For the $N=Z$ nuclei, this minimal value is vanishing and therefore the lowest mode becomes spurious or in other words saying a new symmetry is open. The new symmetry corresponds to the restriction $C_{jpn}^{(2)} = 0$, i. e. the average of the third component of the isospin operator is vanishing. This means that the system is invariant to rotations around any axes in the (X, Y) plane of the isospin space associated to the (jpn) orbits.
- 5) Important quantitative effects are expected for heavy nuclei having the proton and neutron Fermi energies lying far apart from each other.
- 6) The presence of the additional states influences also the structure of the states lying close to those predicted by the standard RPA. Indeed, the actual normalization condition for the

phonon amplitudes implies new values for the X and Y weights. Consequently, the strengths for β^- and β^+ transitions are shared by the "old"-lying close to the standard RPA states- and the "new" states- for which the amplitudes Z are dominant.

7) The standard RPA approach is based on the quasi-boson approximation and therefore it ignores some important dynamic effects (only the terms $A^\dagger A^\dagger, A^\dagger A, AA$ are considered in an approximative manner) and moreover the Pauli principle is violated. By contrast, within the *frn – RPA* all the terms of the model Hamiltonian are taken into account. Also the Pauli principle is, to a certain extent, restored. Due to this feature, large corrections to the double beta transition amplitude as well as to the Ikeda sum rule are expected by changing the RPA to the *frn-RPA*.

8) The equations of motion for the A^\dagger, A and B^\dagger, B operators are coupled by the terms $A^\dagger B^\dagger, A^\dagger B, AB^\dagger, AB$ involved in the quasiparticle Hamiltonian. These terms are multiplied by the factors $U_p V_n U_{p'} U_{n'}, U_p V_n V_{p'} V_{n'}, V_p U_n U_{p'} U_{n'}, V_p U_n V_{p'} V_{n'}$ in the $ph - ph$ interaction (the χ term) and by $U_p U_n U_{p'} V_{n'}, U_p U_n V_{p'} U_{n'}, V_p V_n U_{p'} V_{n'}, V_p V_n V_{p'} U_{n'}$ in the $pp - hh$ interaction (the χ_1 term), respectively. Note that the coupling terms change the number of either proton or neutron quasiparticles by two units. The terms bringing the main contribution to the equations of motion for the operators A^\dagger, A commute with $\hat{N}_{jp} - \hat{N}_{jn}$ but not with $\hat{N}_{jp} + \hat{N}_{jn}$. By contrary the terms having the dominant contribution to the equations of motion for the operators B^\dagger, B commute with $\hat{N}_{jp} + \hat{N}_{jn}$ and not with $\hat{N}_{jp} - \hat{N}_{jn}$. None of the two operators, $\hat{N}_{jp} - \hat{N}_{jn}, \hat{N}_{jp} + \hat{N}_{jn}$, commutes with the coupling terms. Retaining from the χ -interaction the $pp - hh$ terms (those multiplied by $U_p U_n V_{p'} V_{n'}$) and from the χ_1 interaction only the $ph - ph$ terms (those proportional to $U_p V_n U_{p'} V_{n'}$) the equations of motion for the operators B^\dagger, B are decoupled from those for A^\dagger and A . One may conclude that the new mode is determined by a combined effect coming from the $pp - hh$ and $ph - ph$ terms belonging to the χ and χ_1 interactions, respectively.

9) In the particle representation the *frn – RPA* phonon operator is a linear superposition of ph, hp, pp and hh operators.

10) In the limit of large pp and negligible ph interactions, the amplitudes Z can be analyti-

cally calculated. The result is that Z is proportional to either $U_p V_n$ or $V_p U_n$, depending on whether the sign of $E_p - E_n$ is plus or minus, respectively. When this amplitude prevails over the other ones, the corresponding mode describes a neutron-hole proton-particle (or a proton-hole neutron-particle) excitation of the mother nucleus (N, Z) . Therefore the state is associated to the $(N - 1, Z + 1)$ (or to the $(N + 1, Z - 1)$) nucleus. In this case the state might be reached by exciting the ground state through the transition operator $c_p^\dagger c_n$ (or $c_n^\dagger c_p$), which is typical for the β^- (or β^+) decay. Since the double beta decay is conceived as taking place through two successive β^- transitions, one expects that this process is also influenced by considering this new state as an intermediate state characterizing the odd-odd neighboring nucleus.

11) When the pp interaction is small the amplitude Z is proportional to $U_p U_n$ if $E_p > E_n$ or to $V_p V_n$ in the case $E_p < E_n$. The new mode characterizes the nucleus $(N + 1, Z + 1)$ in the first case and the nucleus $(N - 1, Z - 1)$ in the second situation. The transition operators which could excite these states are obviously of the types $c_p^\dagger c_n^\dagger$ and $c_n c_p$, respectively.

Note that the restriction of the phonon operator to the scattering terms resembles the standard RPA boson operator written in the particle representation. This comparison has, however, only a formal value since in the quasiparticle representation there is no Fermi energy and therefore one cannot speak about quasiparticle-quasihole excitations. Similar features are met in solid state physics for the description of electron excitations in narrow energy bands, spin waves and plasma oscillations [13]. In nuclear physics, the scattering terms have been also considered but not for proton-neutron excitations. Indeed, using the thermal response theory, Tanabe [14] studied the charge conserving phonons in nuclear systems at a finite temperature. It seems that the contribution of the scattering terms to the charge conserving bosons, does not survive at vanishing temperature [15]. Moreover, the dispersion relation for the mode energy cannot be obtained from a linearized set of equations as it is required by the spirit of the RPA approach.

III. SEMI-CLASSICAL TREATMENT

As we already mentioned, the scope of the present paper is to study the pn mode caused by the quasiparticle scattering terms within a semi-classical approach. In this formalism the renormalization condition (2.6) is missing and therefore the harmonic motion of the new degrees of freedom hinges on a more physical ground. Moreover, we address the question whether this mode survives when the non-scattering terms are switched off. Thus, it is worth to know if such a mode appears only when the scattering terms accompany the two quasiparticle operators or it might be determined by the scattering terms alone.

From the brief presentation of the previous Section it is clear that the mode does not appear within the RPA approach. Indeed, it occurred within the $frn - RPA$ after a consistent renormalization was performed (i. e. not only the operators A^\dagger, A where renormalized but also B^\dagger and B). If that mode is a signature of the higher RPA formalisms, then it should also appear within the semi-classical formalism as well as in the boson expansion framework. As we shall see the semi-classical approach is able to predict the mode even in the harmonic approximation, the mode being associated with the small oscillations of the system around a static correlated ground state. Moreover, the semi-classical frame is expected to allow us an intuitive interpretation of this new type of excitation.

We recall that the higher order corrections to the standard RPA approach are frequently studied, with different purposes, using a single j case and ignoring the scattering terms. The procedure has the advantage that the resulting Hamiltonian is exactly solvable. Therefore the quality of the adopted approximations may be tested by comparing the predictions with the corresponding exact results.

To touch the goal of the present paper we adopt a similar point of view. Indeed, if the coupling terms (mentioned at the point 8 of the previous section) are ignored, the equations of motion for the scattering operators are decoupled. Moreover the motion of these operators is determined also by an exactly solvable Hamiltonian, which reads:

$$H_{pn}^{(q)} = E'_p \hat{N}_p + E'_n \hat{N}_n + \lambda_1 B^\dagger(pn) B(pn) + \lambda_2 (B^{\dagger 2}(pn) + B^2(pn)), \quad (3.1)$$

where the following notations have been used:

$$\begin{aligned}
E'_p &= E_p + (\chi_1 - \chi) V_p^2, \\
E'_n &= E_n + (\chi + \chi_1) V_p^2 (V_n^2 - U_n^2), \\
\lambda_1 &= \chi (U_p^2 U_n^2 + V_p^2 V_n^2) - \chi_1 (U_p^2 V_n^2 + V_p^2 U_n^2), \\
\lambda_2 &= -(\chi + \chi_1) U_p U_n V_p V_n,
\end{aligned} \tag{3.2}$$

$$\hat{N}_\tau = \sum_m a_{\tau jm}^\dagger a_{\tau jm}. \tag{3.3}$$

Also, to simplify the notation we omitted the quantum number j for the operators $B^\dagger(jpn), B(jpn)$ as well as for the U, V coefficients and quasiparticle energies. This model Hamiltonian will be studied within a time dependent variational formalism. Therefore, some static and dynamic properties will be described by solving the equations provided by the time dependent variational principle (TDVP)¹:

$$\delta \int_0^t \langle \Psi | H_{pn}^{(q)} - i \frac{\partial}{\partial t'} | \Psi \rangle dt' = 0. \tag{3.4}$$

If the variational state $|\Psi\rangle$ spans the whole Hilbert space describing the many-body system, solving the equation (3.4) is equivalent to solving the time dependent Schrödinger equation, which would be a very difficult task. In the present paper, the trial function is taken as:

$$|\Psi\rangle = \exp[z B^\dagger(pn) - z^* B(pn)] |NT - T\rangle, \tag{3.5}$$

where $|NTT_3\rangle$ denotes the common eigenfunction of the quasiparticle total number (\hat{N}), the quasiparticle isospin squared (\hat{T}^2), and its z-axis projection (T_z) operators, respectively. z is a complex function of time and z^* , the corresponding complex conjugate function. We justify this choice by the symmetry properties of the model Hamiltonian. Indeed, let us note first that $H_{pn}^{(q)}$ commutes with the quasiparticle total number operator. Moreover, it can be written in terms of the quasiparticle total number operator and generators of the $SU(2)$ isospin algebra

¹Throughout this paper the units of $\hbar = 1$ are used

$$\begin{aligned}
\tau_{+1} &= -\frac{1}{\sqrt{2}}B^\dagger(pn), \\
\tau_{-1} &= \frac{1}{\sqrt{2}}B(pn), \\
\tau_0 &= \frac{1}{2}(\hat{N}_p - \hat{N}_n).
\end{aligned} \tag{3.6}$$

Due to this property of $H_{pn}^{(q)}$, the function $|\Psi\rangle$, which is a coherent state for the SU(2) group, is the most suitable for a semi-classical treatment.

Before closing this section we would like to write the trial function in a form which suits better the further purposes. Using the Cambel Hausdorff factorization [16] for the exponential function, as explained in Appendix A, one obtains:

$$\begin{aligned}
|\Psi\rangle &= \mathcal{N}e^{\alpha B^\dagger(pn)}|NT - T\rangle, \\
\mathcal{N} &= (1 + \alpha^* \alpha)^{-T}.
\end{aligned} \tag{3.7}$$

where α depends on the polar coordinates, $z = \rho e^{i\varphi}$:

$$\alpha = \tan(\rho)e^{i\varphi}. \tag{3.8}$$

IV. EQUATIONS OF MOTION

In order to write the equations of motion provided by the TDVP (3.4), we need the matrix element of $H_{pn}^{(q)}$ as well as of the time derivative operator, $\frac{\partial}{\partial t}$. These can be evaluated by direct calculation, using the expressions (3.5) when the average of $H_{pn}^{(q)}$ is considered and (3.7) for the classical action. The result is:

$$\begin{aligned}
\langle \Psi | H_{pn}^{(q)} | \Psi \rangle &= -T(E'_p - E'_n + 2\lambda_1) + \frac{N}{2}(E'_p + E'_n) + 2T(E'_p - E'_n + 2\lambda_1) \frac{\alpha^* \alpha}{1 + \alpha^* \alpha} \\
&\quad + 2T(2T - 1) \left[\lambda_1 \frac{\alpha^* \alpha}{(1 + \alpha^* \alpha)^2} + \lambda_2 \frac{\alpha^{*2} + \alpha^2}{(1 + \alpha^* \alpha)^2} \right], \\
\langle \Psi | \frac{\partial}{\partial t} | \Psi \rangle &= T \frac{\alpha^* \dot{\alpha} - \dot{\alpha}^* \alpha}{1 + \alpha^* \alpha}.
\end{aligned} \tag{4.1}$$

Considering α, α^* as classical phase space coordinates, the TDVP equation (3.4) yields the following classical equations of motion, describing the nuclear system:

$$\begin{aligned}\frac{\partial \mathcal{H}}{\partial \alpha} &= -2i \frac{T \dot{\alpha}^*}{(1 + \alpha^* \alpha)^2}, \\ \frac{\partial \mathcal{H}}{\partial \alpha^*} &= 2i \frac{T \dot{\alpha}}{(1 + \alpha^* \alpha)^2}.\end{aligned}\tag{4.2}$$

Here \mathcal{H} denotes the classical energy function:

$$\mathcal{H} = \langle \Psi | H_{pn}^{(q)} | \Psi \rangle.\tag{4.3}$$

In order to quantize the classical trajectories satisfying the equations (4.2) as well as to have an one to one correspondence between the classical and quantal behaviors of the nucleon system, it is convenient to chose those conjugate variables which bring the equations of motion in a canonical Hamilton form. A possible choice of the coordinates with the above mentioned property is

$$r = \frac{2T}{1 + \alpha^* \alpha},\tag{4.4}$$

$$\psi = -\frac{1}{2i}(\ln \alpha - \ln \alpha^*) = -\varphi.\tag{4.5}$$

Indeed, in the new variables the classical equations read:

$$\begin{aligned}\frac{\partial \mathcal{H}}{\partial r} &= -\dot{\psi}, \\ \frac{\partial \mathcal{H}}{\partial \psi} &= \dot{r}.\end{aligned}\tag{4.6}$$

with the classical energy:

$$\begin{aligned}\mathcal{H} &= T(E'_p - E'_n + 2\lambda_1) + \frac{N}{2}(E'_p + E'_n) \\ &\quad - (E'_p - E'_n + 2\lambda_1)r + \frac{2T-1}{2T}r(2T-r)(\lambda_1 + 2\lambda_2 \cos 2\psi).\end{aligned}\tag{4.7}$$

Note that r has the significance of a generalized coordinate while ψ that of generalized linear momentum. Due to the generalized momentum ψ , the equations motion are not linear and therefore analytical solutions are not obtainable. The equations can however be approximatively solved if they are linearized around the minimum point of the energy function:

$$\overset{\circ}{r} = T \left[1 - \frac{E'_p - E'_n + 2\lambda_1}{(2T-1)(\lambda_1 - 2\lambda_2)} \right], \quad \overset{\circ}{\psi} = \frac{\pi}{2}. \quad (4.8)$$

In order that the minimum exists, it is necessary that the generalized coordinates satisfy a consistency condition, required by the definition range of r :

$$0 \leq \overset{\circ}{r} \leq 2T. \quad (4.9)$$

By means of (4.8), this provides a constraint for the strengths of the two body interactions. The linearized equations, written in terms of the deviations

$$q = r - \overset{\circ}{r}, \quad p = \psi - \overset{\circ}{\psi}, \quad (4.10)$$

are of harmonic type:

$$\begin{aligned} -\overset{\bullet}{p} &= -2 \frac{2T-1}{2T} (\lambda_1 - 2\lambda_2) q, \\ \overset{\bullet}{q} &= 4 \frac{2T-1}{T} \overset{\circ}{r} (2T - \overset{\circ}{r}) \lambda_2 p. \end{aligned} \quad (4.11)$$

These describe a harmonic motion for the conjugate coordinates, with the angular frequency:

$$\omega = 2 \frac{2T-1}{T} [-\lambda_2(\lambda_1 - 2\lambda_2) \overset{\circ}{r} (2T - \overset{\circ}{r})]^{\frac{1}{2}}. \quad (4.12)$$

The condition that ω is a real quantity brings an additional constraint for the strength parameters χ, χ_1 :

$$\chi \geq \chi_1 \left(\frac{U_p V_n - V_p U_n}{U_p U_n + V_p V_n} \right)^2. \quad (4.13)$$

As we said already before, the schematic model has the advantage, over the realistic formalisms, that allows us to compare the approximative solutions with the exact one. For the particular Hamiltonian used in the present paper, the exact eigenvalues can be obtained by diagonalization in the basis $|NTM\rangle$. Indeed, in this basis the model Hamiltonian has the following non-vanishing matrix elements:

$$\begin{aligned} \langle NTM | H_{pn}^{(q)} | NTM \rangle &= \frac{1}{2} (E'_p + E'_n) N + (E'_p - E'_n) M + \lambda_1 (T + M)(T - M + 1), \\ \langle NTM + 2 | H_{pn}^{(q)} | NTM \rangle &= \lambda_2 [(T - M - 1)(T - M)(T + M + 1)(T + M + 2)]^{\frac{1}{2}}, \\ \langle NTM | H_{pn}^{(q)} | NTM + 2 \rangle &= \langle NTM + 2 | H_{pn}^{(q)} | NTM \rangle. \end{aligned} \quad (4.14)$$

V. THE RENORMALIZED RPA AND BOSON EXPANSION

Within the RPA approach, the renormalization of the quasiparticle mean field due to the two quasiparticle interactions is usually ignored. Therefore the Hamiltonian considered is:

$$H_{qp} = E_p \hat{N}_p + E_n \hat{N}_n + \lambda_1 B^\dagger(pn) B(pn) + \lambda_2 (B^{\dagger 2}(pn) + B^2(pn)), \quad (5.1)$$

The operators B^\dagger, B satisfy the commutation relation:

$$[B(pn), B^\dagger(pn)] = \hat{N}_n - \hat{N}_p. \quad (5.2)$$

If the r.h. side of the above equation is replaced by its average on the ground state,

$$C = \langle 0 | \hat{N}_n - \hat{N}_p | 0 \rangle \quad (5.3)$$

which is to be determined, then the operators B, B^\dagger become bosons, after the following renormalization

$$\tilde{B}^\dagger(pn) = \frac{1}{\sqrt{C}} B^\dagger(pn), \quad \tilde{B}(pn) = \frac{1}{\sqrt{C}} B(pn), \quad (5.4)$$

if C is positive, while for negative C the renormalized operators are:

$$\tilde{B}^\dagger(pn) = \frac{1}{\sqrt{|C|}} B(pn), \quad \tilde{B}(pn) = \frac{1}{\sqrt{|C|}} B^\dagger(pn). \quad (5.5)$$

Suppose, for the time being, that $C > 0$. If that is not the case the corresponding calculations can be worked out in a similar way. The equations of motion for the renormalized operators are:

$$\begin{aligned} [H_{qp}, \tilde{B}^\dagger(pn)] &= (E_p - E_n + \lambda_1 C) \tilde{B}^\dagger(pn) + 2\lambda_2 C \tilde{B}(pn), \\ [H_{qp}, \tilde{B}(pn)] &= -2\lambda_2 C \tilde{B}^\dagger(pn) - (E_p - E_n + \lambda_1 C) \tilde{B}(pn). \end{aligned} \quad (5.6)$$

Since the equations are linear in $\tilde{B}^\dagger(pn)$ and $\tilde{B}(pn)$, one can define the phonon operator

$$\Gamma^\dagger = X \tilde{B}^\dagger(pn) - Y \tilde{B}(pn), \quad (5.7)$$

with the amplitudes determined such that the following equations are fulfilled:

$$\begin{aligned} [H_{qp}, \Gamma^\dagger] &= \omega \Gamma^\dagger, \\ [\Gamma, \Gamma^\dagger] &= 1. \end{aligned} \quad (5.8)$$

The first equation provides the dispersion equation for the mode energy

$$\omega = \left[(E_p - E_n + \lambda_1 C)^2 - 4\lambda_2^2 C^2 \right]^{\frac{1}{2}}, \quad (5.9)$$

while the second one the normalization relation for phonon amplitudes:

$$X^2 - Y^2 = 1. \quad (5.10)$$

The renormalized RPA vacuum is defined by

$$\Gamma|0\rangle = 0. \quad (5.11)$$

The solution of the above equation is:

$$|0\rangle = e^{-\frac{1}{8}(\frac{Y}{X})^2} e^{\frac{Y}{2X} \tilde{B}^2} |NT, -T\rangle. \quad (5.12)$$

Then the renormalization constant C can be exactly evaluated:

$$C = 2T - 2 + \frac{2}{X^2}. \quad (5.13)$$

Since $T \geq 1$, the constant C is always positive. The equations of motion allow us to express the amplitude Y in terms of X :

$$Y = \frac{1}{2\lambda C} [\omega - (E_p - E_n + \lambda_1 C)] X, \quad (5.14)$$

which together with the normalization condition (5.10) determines fully the amplitudes X and Y in terms of C and ω . Inserting the result for X into the equation (5.13), one obtains an equation for C as a function of ω . This and eq.(5.9) form a set of two nonlinear equations for the unknowns ω and C .

As we mentioned before, another way to improve the RPA treatment is to use the boson expansion concept. Through this procedure, the $SU(2)$ algebra, with the fermionic generators $\tau_{\pm 1}, \tau_0$ defined by eq.(3.6), is mapped to a boson $SU(2)$ algebra, generated by $\hat{T}_{\pm 1}, \hat{T}_0$. Denoting by b^+, b a pair of boson operators, the $SU(2)$ algebra generators $\hat{T}_{\pm 1}, \hat{T}_0$ can be constructed as function of b^+ and b . The resulting expressions are conventionally called as the boson expansion of the fermionic generators, respectively. There are three distinct boson mappings for the fermionic $SU(2)$ algebra found by Holstein-Primakoff [17], Dyson [18] and one of the present authors (A. A. R.) [19], respectively. For the present purpose here we use the Holstein-Primakoff (HP) expansion:

$$\begin{aligned}\hat{T}_{+1} &= -\sqrt{T}b^+ \left(1 - \frac{b^+b}{2T}\right)^{\frac{1}{2}}, \\ \hat{T}_{-1} &= \sqrt{T} \left(1 - \frac{b^+b}{2T}\right)^{\frac{1}{2}} b, \\ \hat{T}_0 &= b^+b - T.\end{aligned}\tag{5.15}$$

By a direct calculation it can be checked that, by this mapping, to the operator τ^2 it corresponds a C-number:

$$\hat{T}^2 = T(T+1).\tag{5.16}$$

The fermion Hamiltonian H_{qp} commutes with the quasiparticle total number and the same is true for the generators $\tau_{\pm 1}, \tau_0$. Therefore the image of the quasiparticle total number operator through the HP mapping is invariant against any rotation in the isospin space and consequently, according to the above equation, is a C-number. Apart from an additive constant, the image of H_{qp} through the HP boson expansion is:

$$H_{qp}^{(b)} = (E_p - E_n)\hat{T}_0 - 2\lambda_1\hat{T}_{+1}\hat{T}_{-1} + 2\lambda_2(\hat{T}_{+1}^2 + \hat{T}_{-1}^2).\tag{5.17}$$

Making use of eqs. (5.15), the boson mapping of H_{qp} is a infinite series in the bosons b^+, b , due to the square root operators. Expanding the square root operators and truncating the result at the second order in bosons, the boson Hamiltonian becomes:

$$H_{qp;2}^{(b)} = (E_p - E_n + 2\lambda_1 T)b^+b + 2\lambda_2 T(b^{+2} + b^2). \quad (5.18)$$

For a limited range of the interaction strength, this Hamiltonian can be diagonalized through a canonical transformation:

$$\begin{aligned} b^+ &= UB^+ + VB, \\ b &= UB + VB^+, \\ 1 &= U^2 - V^2. \end{aligned} \quad (5.19)$$

The restriction that the "dangerous" terms have a vanishing strength yields the expression for the transformation coefficients and the coefficient, ω_1 , of the diagonal term B^+B :

$$\begin{aligned} \begin{pmatrix} U \\ V \end{pmatrix} &= \frac{1}{\sqrt{2}} \left[\mp 1 + \frac{|E_p - E_n|}{\sqrt{(E_p - E_n + 2\lambda_1 T)^2 - 16\lambda_2^2 T^2}} \right]^{\frac{1}{2}}, \\ \omega_1 &= \left[(E_p - E_n + 2\lambda_1 T)^2 - 16\lambda_2^2 T^2 \right]^{\frac{1}{2}}. \end{aligned} \quad (5.20)$$

Comparing the expressions of ω_1 (5.20) and ω (5.9), one sees that the two energies are identical for the limiting case of $X = 1$, which is met when $\lambda_2 = 0$ (see eqs. (5.9) and (5.14)).

At this stage it is worthwhile to make the following remarks: a) When the HP boson expansion of the model Hamiltonian is truncated at the second order terms in bosons, the quasiparticle total number operator is no longer a C number. Therefore the contribution of this term should have been considered in a consistent manner. Moreover the truncation is justified only for large values of the total isospin T. b) The same inconsistency appears in the calculation of the renormalization constant C. Indeed the expression (5.14) is exact and therefore includes all contributions coming from the infinite boson series of the correlated ground state, given by (5.14). c) Since the boson mapping (5.15) is an unitary transformation, the exact eigenvalues of H_{qp} are reproduced by diagonalizing the boson expanded Hamiltonian $H_{qp}^{(b)}$. For the second order truncated Hamiltonian, the canonical transformation breaks down at a critical value of the attractive interaction strength. However the diagonalization procedure is able to find the eigenvalues for any strength of the attractive

interaction. The resulting energies exhibits a phase transition (the first derivative has a jump) at the critical value of the strength. If the second branch of the energy curve could also be approximated by an harmonic mode, describing small oscillations of the classical system around a stationary state, this is still an open question [12]. d) The HP boson representation provides for the harmonic mode the interpretation of an wobbling motion of the system around the total isospin. e) The HP boson expansion is justified (in the sense that some eigenvalues of the truncated Hamiltonian are close to the corresponding exact ones) when the rotation axis in the isospin space is close to the quantization axis (z axis), which is usually taken as the axis to which the maximum “moment of inertia” corresponds. If the angle between the rotation axis and z-axis is large the harmonic energy may collapse. In this case the quantization axis should be chosen as one of the X and Y axes depending on the magnitude of the strength of the τ_x^2 and τ_y^2 terms from the quasiparticle Hamiltonian $H_{pn}^{(q)}$. In this case the boson representation suitable for the low order description should be of Dyson type [20]. The harmonic approximation for the new representation describes also a wobbling motion of a frequency equal to the square root of the product of the inverse of the non-maximal moments of inertia normalized to the inverse of the maximal moment of inertia.

VI. NUMERICAL RESULTS

The formalism described in the previous sections, has been applied to the case $j = \frac{19}{2}$. On the proton level, 6 protons are distributed while in the neutron level, 14 neutrons. Alike nucleons interact with each other through pairing forces whose strength are $G_p = 0.2\text{MeV}$ and $G_n = 0.4\text{ MeV}$. From the pairing equations it results the following expression for the quasiparticle energy:

$$E_\tau = \frac{1}{2}G_\tau\Omega, \quad \Omega = \frac{2j+1}{2}. \quad (6.1)$$

With the data specified above the result for the quasiparticle energy is:

$$E_p = 2 \text{ MeV}, \quad E_n = 1 \text{ MeV}. \quad (6.2)$$

According to our previous study, the renormalized RPA ground state involves a small number of quasiparticles. For example, for a small strength of the particle-particle interaction, the quasiparticle total number is about 2 while for large values of the above mentioned strength the number may reach the value 4. Due to this behavior of the correlated ground state we considered for the isospin carried by the quasiparticles in the ground state, alternatively the values 1 and 2. Although these values vary with increasing the particle-particle strength we kept them constant.

The numerical analysis refers to the dependence of the energy ω of the new nuclear mode, on the strengths of the ph and pp monopole interactions, χ, χ_1 . Aiming at showing how good is the semi-classical approach for this new type of pn excitation, we calculated also the exact eigenvalues of the model Hamiltonian, by diagonalizing the associated matrix (4.14) within the basis $|NTM\rangle$. The results are shown in Fig. 1 and Fig. 2. From Fig. 1, one notices that the harmonic mode collapses for a critical value of the attractive interaction χ_1 . This critical value is certainly depending on the repulsive interaction strength. The larger is that strength the larger the critical value. In Fig. 1, we have also plotted the normalized energy for the first excited state. There are intervals for χ_1 where the energy of the harmonic mode approximates reasonably well the exact excitation energy. Moreover, for two values of the strength parameter, the exact solutions are precisely reproduced. For the case $T=2$, the two energies, exact and ω , are the same for $\chi_1 = 0$ and $\chi_1 = 0.3$ for $\chi = 1$ and $\chi = 0.5$ respectively, but the curves are going apart for the first part of interval and then converge to an intersection point close to the critical value.

The peculiar feature of ω as a function of χ_1 , which distinguishes it from the standard RPA modes, consists of its non-monotonic behavior with respect to the increase of the strength of the pp interaction. The reason is that in the common cases the mean field is constant when the two-body interaction is varied, while here by changing χ_1 we change also the minimum point for energy and therefore another mean field is obtained. It is interesting

to notice that although the ph interaction, the χ term, is kept constant, the change of the mean field is equivalent to an increase of the effective ph interaction until ω reaches the maximum value from where the attractive component of the two body interaction prevails.

In Fig. 2, the energies ω and the normalized energy of the first excited state are shown as function of χ , the strength parameter of the ph interaction. Both energies are monotonically increasing with the increase of the interaction strength. In contrast to what happens in the case of χ_1 dependence, here the change of the mean field by changing the energy minimum does not change the repulsive character of the χ interaction. The agreement between ω and the exact energy of the first excited state is reasonable good.

In Fig. 3. the energies characterizing the harmonic mode predicted by the renormalized RPA and semi-classical method are plotted as function of χ_1 . Also, the exact energy of the first excited state is presented. Although they have different trends, the semi-classical and renormalized RPA energies are not far from each other for $\chi < 0.45$. At the critical value $\chi=0.57$ the energy yielded by the renormalized RPA is going very fast to zero. This behavior is specific to the present model where only the scattering terms are considered. Indeed, if the phonon operator includes both the two quasiparticle and scattering terms, the corresponding mode collapses for larger χ_1 . In the semi-classical treatment this happens only for very large χ_1 since the static ground state is changed by increasing χ_1 . The result obtained with the truncated HP boson expanded Hamiltonian (see eq. (5.20)) is very close to the result shown in Fig. 3 for the renormalized RPA procedure.

Comparing the results from Fig. 1a and Fig. 3, we remark on the following features. While the renormalized RPA energy collapses at a relatively small value of χ_1 , the mode energy predicted by the semi-classical formalism vanishes for a very large χ_1 , far beyond the realistic value, which is $\chi_1 = \chi$. This feature is a consequence of changing the static ground state with χ_1 . The energy behavior provided by the semi-classical method is also different from that predicted by the standard renormalized pnQRPA (see for example ref. 9) where the mode energy is a monotonic function of χ_1 and goes asymptotically to zero.

In this context we recall that the *frn* – RPA breaks down [10] before the standard

RPA does, and that happened due to the fact that the lowest $frn - RPA$ energy is that associated with the new collective mode. From the present calculations one sees that *this is not true within the semi-classical approach and therefore including the scattering terms in the expression of the phonon operator does not prevent the treatment of the many-body system for a realistic value of the pp-interaction strength.*

The vanishing energies for the new mode, shown in Figs. 1a and 3 suggest that a phase transition occurs according to the corresponding formalisms. As we already mentioned this is clearly revealed if one diagonalizes the Hamiltonian given by eq. (5.18) [11,12]. In the renormalized RPA procedure the new phase is determined by a new minimum of the classical energy associated to $H_{pn;2}^{(b)}$, reflecting the fact that the λ_2 term is the dominant one for these values of χ_1 . In the full-line and dotted-line curves of Fig. 1a, the corresponding energies also vanish at certain critical values which result in having again a phase transition. This is reflected in the curve obtained by exact calculations, by the fact that the energy is minimum for the critical strength. The increasing branch shown by the exact calculations (corresponding to the second nuclear phase) might be semi-classically described by changing the trial function, involved in the time dependent variational equations, by rotating it (in the isospin space) with an angle which corresponds to the orientation of the axis of maximum “moment of inertia”.

It is remarkable that the far intersection points of the curves obtained by semi-classical and exact calculations respectively, are lying close to the critical values of the semi-classical description. Also, the first intersection point is not far from the critical value of the renormalized RPA treatment. In the classical treatment this feature is well known [21]. Indeed, in the above quoted reference it is shown, for a triaxial rotor cranked on an arbitrarily oriented axis, that for certain critical values of the strength parameters, the period of the harmonic orbits is equal to the period characterizing the motion on the closed exact orbit.

VII. CONCLUSIONS

The main result of this paper refers to the existence of an harmonic mode determined by the scattering quasiparticle terms, which are usually neglected in the standard RPA approach.

The new mode is described within a time dependent variational formalism with an exactly solvable many-body Hamiltonian. The variational state is a coherent state for the underlying symmetry group, which is the SU(2) group. A pair of classical canonical conjugate coordinates, which bring the equations of motion to the Hamilton form, is found. The classical energy has an interesting structure. It is quadratic in coordinate but highly non-linear in the conjugate momentum. Therefore one finds first the stationary point which minimizes the energy, and then linearizes the equations of motion around the minimum point in the classical phase space. The solution for the linearized equations is harmonic and its time period determines the energy of the new mode. Despite the fact the classical system has an harmonic motion, the mode does not exist in the standard RPA approach. In this sense one may say that the present description corresponds to a "renormalized RPA". However as we have seen, by comparing the corresponding predictions, the renormalization involved in the semi-classical description is completely different from the renormalization described in Section V as well as from the boson expansion method.

It is known the fact that the topological structure of the energy surface depends on the strength parameters involved in the model Hamiltonian. Thus, in the parameters space one can define several regions, each of them corresponding to a distinct nuclear phase. Having this in mind, we studied the behavior of the new mode energy when the strength parameter for the *pp* interaction (χ_1) is varied. A particular feature for the semi-classical description is that the energy is not monotonic decreasing function of χ_1 , but it increases in the first part of the interval, reflecting that here the *ph* two body interaction prevails, reaches a maximum value, then decreases and finally vanishes. This property is caused by that for each χ_1 a new ground state is determined. This aspect is missing in both the renormalized RPA and

boson expansion procedures.

Since the model Hamiltonian resembles the triaxial rotor which was semi-classically studied by one of the present authors (A. A. R) in refs. [20,21], the interpretation of the new mode is imported from there. Thus, the new mode describes a wobbling motion around a given total isospin.

The vanishing energy is a sign for a phase transition. In the first phase the rotation axis, in the isospin space lies close to the z-axis, which has the maximum moment of inertia in the region of small χ_1 , while for χ_1 larger than the critical value (where the energy vanishes), the rotation axis lies closer to the (X,Y) plane in the isospin space. While the first phase may be described by a HP boson expansion formalism, for the second phase the Dyson boson representation is the proper one [20]. In the semi-classical approach, the new phase might be described by changing the trial function associated to the first phase, through a rotation which brings the z-axis to the actual axis of maximal “moment of inertia”.

The occurrence of the phase transition can be noticed also in the curve showing the exact first excitation energy as function of χ_1 . Indeed at the critical value of χ_1 , this curve exhibits a minimum.

Another critical values of χ_1 are those where the mode energy is equal to the exact excitation energy produced by the diagonalization procedure. For these values the linearization does not affect at all the period of the exact closed classical orbit. As a matter of fact, for χ_1 lying close to these points the linearization are best justified. It is interesting to notice that these values of χ_1 lie however close to the values where the phase transitions in the semi-classical treatment (the far intersection point) and the renormalized RPA approach (the near intersection point) take place. This observation allows us to conclude that the semi-classical approach works very well for the values of χ_1 where the renormalized RPA breaks down and that the interval where the linearization procedure does not work, ending with the critical value where the semi-classical energy vanishes, is very narrow.

The energy of the new mode vanishes for a value of χ_1 which is far beyond the physical value ($\chi_1 = \chi$). In this way the drawback of the *frn-RPA*, of breaking down earlier than

the standard RPA does, is removed.

How could the new state be populated? We identified the transition operators which could excite the new state from the ground state. The conclusion is that these state can be seen either in a β^- (or β^+) decay or in a deuteron transfer reaction experiment.

The coupling of this mode to other collective states will be studied in a subsequent paper using a realistic interaction and a large model space for the single particle motion.

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VIII. APPENDIX A

Here to derive the factorization of the trial function $|\Psi\rangle$. To this purpose we address the following more general question. Which are the t-functions $A(t), B(t), C(t)$ satisfying the equation

$$e^{t[(z \sum_m a_{pm}^\dagger a_{nm} - z^* \sum_m a_{nm}^\dagger a_{pm})]} = e^{A(t)a_p^\dagger a_n} e^{C(t)(\hat{N}_p - \hat{N}_n)} e^{B(t)a_n^\dagger a_p}, \quad (8.1)$$

with the initial conditions

$$A(0) = B(0) = C(0) = 0 \quad (8.2)$$

and t a real parameter. Once we solve this problem the needed factorization is obtain from (4.1) for $t = 1$. Taking the first derivative of the eq.(6.1), with respect to t, and identifying the coefficients of the similar operators one obtains the following system of differential equations for the three unknown functions, $A(t), B(t), C(t)$:

$$\dot{z} = \dot{A} - 2A \dot{C} - \dot{B} A^2 e^{-2C(t)},$$

$$\begin{aligned}
0 &= \dot{C} + \dot{B} A e^{-2C(t)}, \\
- \dot{z}^* &= \dot{B} e^{-2C(t)}. \tag{8.3}
\end{aligned}$$

Eliminating the functions B, C from these equations, one obtains the following equation for $A(t)$.

$$\dot{z} = \dot{A} - A^2 z^* \tag{8.4}$$

which admits the solution:

$$A(t) = \tan(\rho t) e^{i\varphi}. \tag{8.5}$$

Here the polar coordinates $(\rho, \varphi)(z = \rho e^{i\varphi})$ have been used. Inserting the result for $A(t)$ in the eq. (6.3), the equations for the remaining functions can be easily integrated. The result is:

$$\begin{aligned}
C(t) &= -\ln(\cos(\rho t)) \\
B(t) &= \tan(\rho t) e^{-i\varphi}. \tag{8.6}
\end{aligned}$$

For the sake of simplifying the writing, hereafter the following notation will be used:

$$\alpha = A(1) \tag{8.7}$$

Using these results the trial function can be written as:

$$|\Psi\rangle = e^{-2C(1)T} e^{A(1)B^\dagger(pn)} |NT - T\rangle \equiv \mathcal{N} e^{A(1)B^\dagger(pn)} |NT - T\rangle \tag{8.8}$$

where \mathcal{N} denotes the normalization factor:

$$\mathcal{N} = e^{-2C(1)T} = e^{2\ln(\cos \rho)T} = (1 + |\alpha|^2)^{-T}. \tag{8.9}$$

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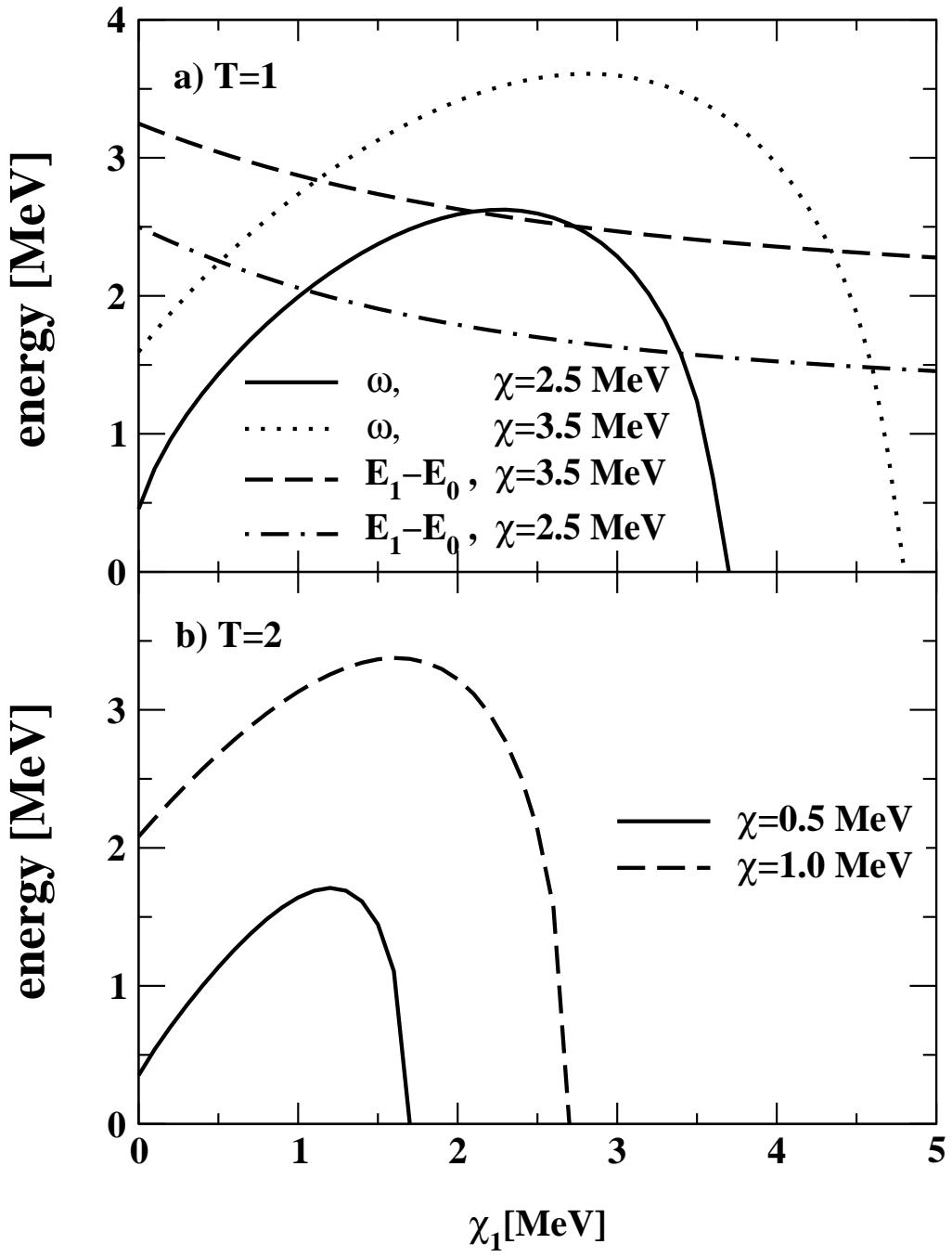


FIG. 1. The energy of the harmonic mode given by the eq. (4.12) and the energy of the first excited state normalized to the ground state are plotted as function of χ_1 for $T = 1$ (a)) and $T = 2$ (b)) and several values of the particle-hole interaction strength, χ .

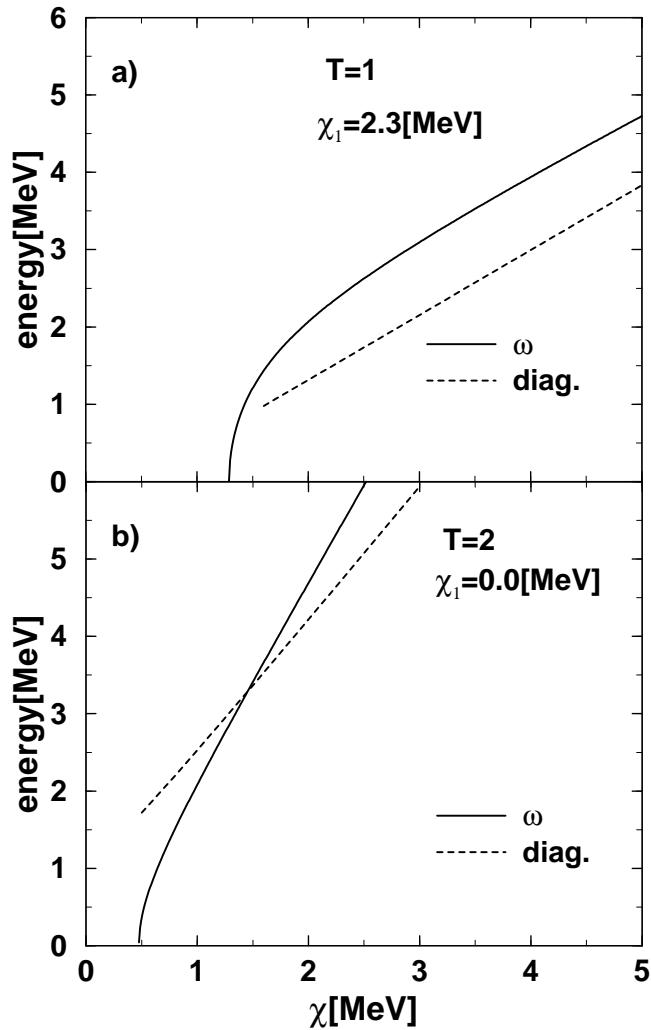


FIG. 2. The energy of the harmonic mode given by the eq. (4.12) and the energy of the first excited state normalized to the ground state are plotted as function of χ_1 for $T = 1, \chi_1 = 2.3 MeV$ (a)) and $T = 2, \chi_1 = 0.0 MeV$ (b)).

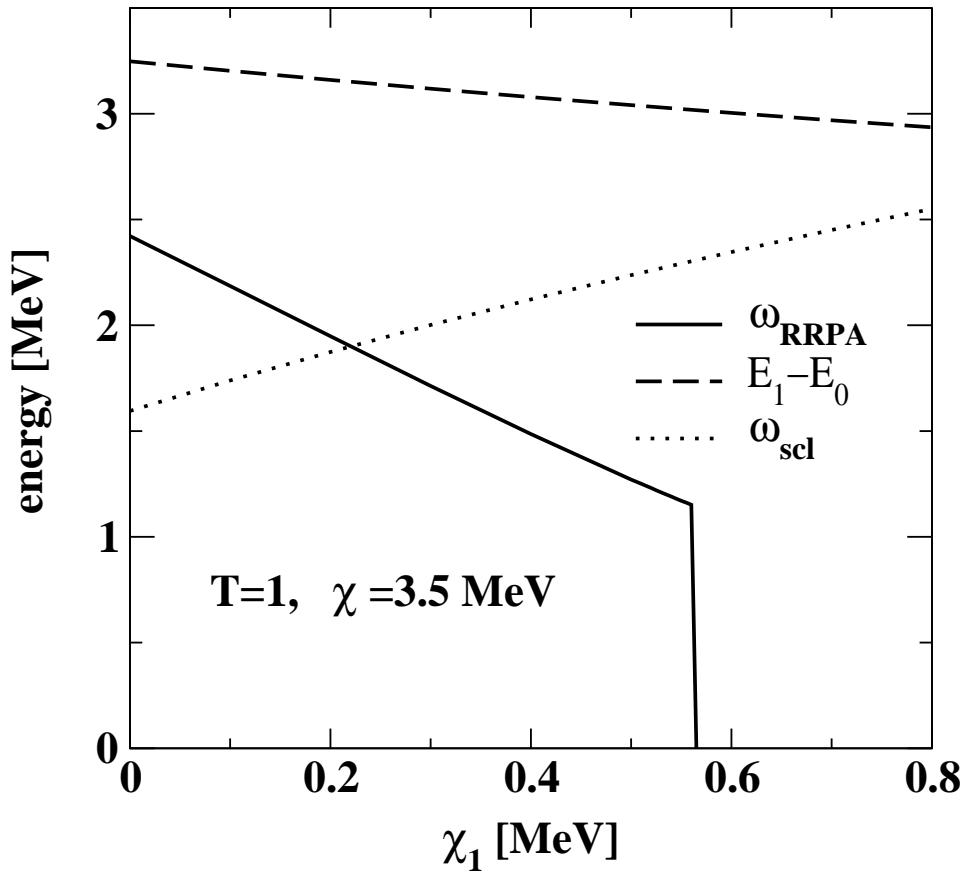


FIG. 3. The energies yielded by the renormalized RPA description (5.9)(full line), semi-classical approach (4.12) (dotted line) and by diagonalization procedure (dashed line) are plotted as function of χ_1 for $T = 1$ and $\chi = 3.5$ MeV.